Math 4550 Homework 2 Solutions

$\boxed{① (a)}$

$\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

$\bar{0}$ has order 1 since its the identity

$$\left.\begin{aligned} &\bar{1} \\ &\bar{1}+\bar{1} = \bar{2} \\ &\bar{1}+\bar{1}+\bar{1} = \bar{3} \\ &\bar{1}+\bar{1}+\bar{1}+\bar{1} = \bar{4} = \bar{0} \end{aligned}\right\} \text{$\bar{1}$ has order 4}$$

$$\left.\begin{aligned} &\bar{2} \\ &\bar{2}+\bar{2} = \bar{4} = \bar{0} \end{aligned}\right\} \text{$\bar{2}$ has order 2}$$

$$\left.\begin{aligned} &\bar{3} \\ &\bar{3}+\bar{3} = \bar{6} = \bar{2} \\ &\bar{3}+\bar{3}+\bar{3} = \bar{9} = \bar{1} \\ &\bar{3}+\bar{3}+\bar{3}+\bar{3} = \overline{12} = \bar{0} \end{aligned}\right\} \text{$\bar{3}$ has order 4}$$

$\boxed{\text{1(b)}}$ $\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$\bar{0}$ has order 1 since it's the identity element

$\bar{1}$

$\bar{1}+\bar{1}=\bar{2}$

$\bar{1}+\bar{1}+\bar{1}=\bar{3}$

$\bar{1}+\bar{1}+\bar{1}+\bar{1}=\bar{4}$

$\bar{1}+\bar{1}+\bar{1}+\bar{1}+\bar{1}=\bar{5}=\bar{0}$

$\bar{1}$ has order 5

$\bar{2}$

$\bar{2}+\bar{2}=\bar{4}$

$\bar{2}+\bar{2}+\bar{2}=\bar{6}=\bar{1}$

$\bar{2}+\bar{2}+\bar{2}+\bar{2}=\bar{8}=\bar{3}$

$\bar{2}+\bar{2}+\bar{2}+\bar{2}+\bar{2}=\overline{10}=\bar{0}$

$\bar{2}$ has order 5

$\bar{3}$

$\bar{3}+\bar{3}=\bar{6}=\bar{1}$

$\bar{3}+\bar{3}+\bar{3}=\bar{9}=\bar{4}$

$\bar{3}+\bar{3}+\bar{3}+\bar{3}=\overline{12}=\bar{2}$

$\bar{3}+\bar{3}+\bar{3}+\bar{3}+\bar{3}=\overline{15}=\bar{0}$

$\bar{3}$ has order 5

$\bar{4}$

$\bar{4}+\bar{4}=\bar{8}=\bar{3}$

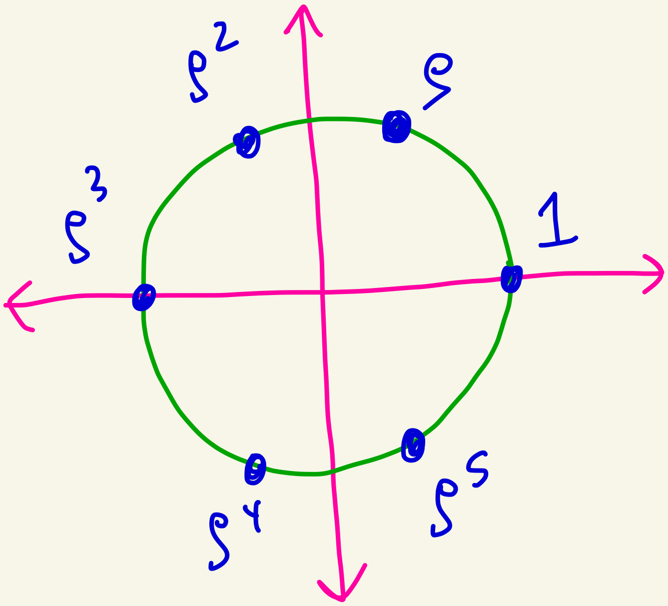
$\bar{4}+\bar{4}+\bar{4}=\overline{12}=\bar{2}$

$\bar{4}+\bar{4}+\bar{4}+\bar{4}=\overline{16}=\bar{1}$

$\bar{4}+\bar{4}+\bar{4}+\bar{4}+\bar{4}=\overline{20}=\bar{0}$

$\bar{4}$ has order 5

② $U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$



where $\rho = e^{\frac{2\pi i}{6}} = e^{\frac{\pi}{3} i}$.

and $\boxed{\rho^6 = 1}$

key formula for below

1 has order 1 since its the identity element

$\rho \neq 1$
$\rho^2 \neq 1$
$\rho^3 \neq 1$
$\rho^4 \neq 1$
$\rho^5 \neq 1$
$\rho^6 = 1$

ρ has order 6

$\rho^2 \neq 1$
$(\rho^2)^2 = \rho^4 \neq 1$
$(\rho^2)^3 = \rho^6 = 1$

So, ρ^2 has order 3.

$\rho^3 \neq 1$
$(\rho^3)^2 = \rho^6 = 1$

So, ρ^3 has order 2

$\rho^4 \neq 1$
$(\rho^4)^2 = \rho^8 = \rho^6 \cdot \rho^2 = \rho^2 \neq 1$
$(\rho^4)^3 = \rho^{12} = \rho^6 \cdot \rho^6 = 1 \cdot 1 = 1$

So, ρ^4 has order 3

$\rho^5 \neq 1$
$(\rho^5)^2 = \rho^{10} = \rho^6 \cdot \rho^4 = \rho^4 \neq 1$
$(\rho^5)^3 = \rho^{15} = \rho^6 \rho^6 \rho^3 = \rho^3 \neq 1$
$(\rho^5)^4 = \rho^{20} = \rho^6 \rho^6 \rho^6 \rho^2 = \rho^2 \neq 1$
$(\rho^5)^5 = \rho^{25} = (\rho^6)^4 \cdot \rho = 1 \cdot \rho = \rho \neq 1$
$(\rho^5)^6 = \rho^{30} = (\rho^6)^5 = 1^5 = 1$

So, ρ^5 has order 6

③ $D_6 = \{1, r, r^2, s, sr, sr^2\}$

Where $r^3 = 1$, $s^2 = 1$, $r^k s = s r^{-k} = s r^{3-k}$

1 has order 1 since its the identity

$r \neq 1$ $r^2 \neq 1$ $r^3 = 1$ So, r has order 3	$r^2 \neq 1$ $(r^2)^2 = r^4 = r^3 \cdot r = 1 \cdot r = r \neq 1$ $(r^2)^3 = r^6 = r^3 \cdot r^3 = 1 \cdot 1 = 1$ So, r^2 has order 3
$s \neq 1$ $s^2 = 1$ so, s has order 2	$sr \neq 1$ $(sr)^2 = srsr = ssr^{-1}r = s^2 \cdot 1 = 1 \cdot 1 = 1$ So, sr has order 2

$sr^2 \neq 1$

$(sr^2)^2 = sr^2sr^2 = ssr^{-2}r^2 = s^2 \cdot 1 = 1 \cdot 1 = 1$

So, sr^2 has order 2

4

$\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \underset{\bar{2}+\bar{2}}{\bar{4}}, \underset{\bar{2}+\bar{2}+\bar{2}}{\bar{6}}\}$

stops at $\bar{6}$ since
$\bar{2}+\bar{2}+\bar{2}+\bar{2} = \bar{8} = \bar{0}$

$\langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$

stops at $\bar{4}$ since
$\bar{4}+\bar{4} = \bar{8} = \bar{0}$

$\langle \bar{5} \rangle = \{\bar{0}, \bar{5}, \bar{2}, \bar{7}, \bar{4}, \bar{1}, \bar{6}, \bar{3}\}$

$\bar{2} = \bar{5}+\bar{5}$

$\bar{7} = \bar{5}+\bar{5}+\bar{5}$

$\bar{4} = \bar{5}+\bar{5}+\bar{5}+\bar{5}$

$\bar{1} = \bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}$

$\bar{6} = \bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}$

$\bar{3} = \bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}+\bar{5}$

$\boxed{⑤}$

$$U_8 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6, \rho^7\}$$

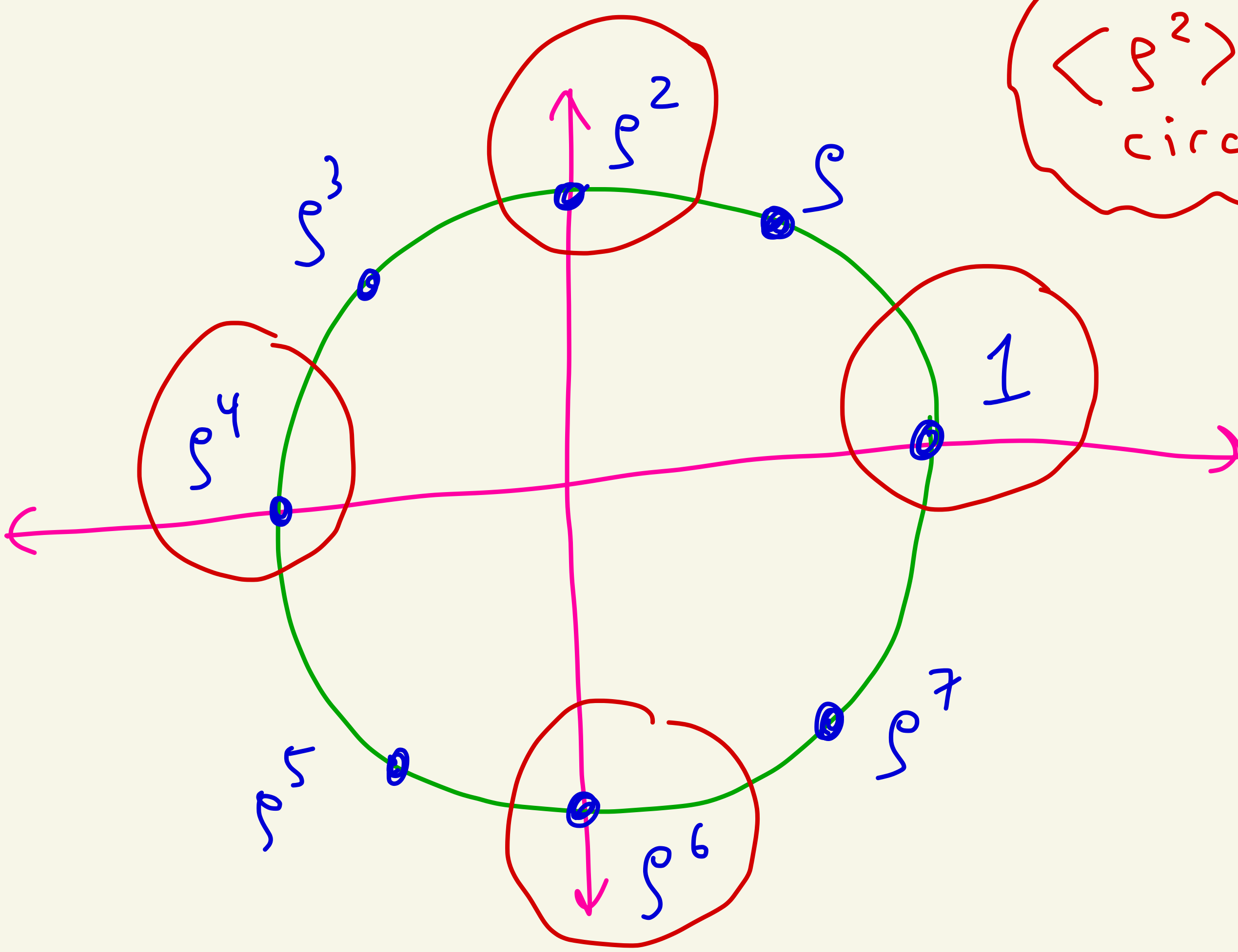
where $\rho = e^{\frac{2\pi i}{8}} = e^{\frac{\pi}{4} i}$ and $\rho^8 = 1$.

We want $\langle e^{2\pi i/4} \rangle = \langle \rho^2 \rangle$

We have

$$\langle \rho^2 \rangle = \{1, \rho^2, \underset{(\rho^2)^2}{\underset{\uparrow}{\rho^4}}, \underset{(\rho^2)^3}{\underset{\uparrow}{\rho^6}}\}$$

stops at ρ^6 since $(\rho^2)^4 = \rho^8 = 1$



$\langle \rho^2 \rangle$ is circled

⑥ $\mathbb{R}^* = \mathbb{R} - \{0\}$ is a group under multiplication.

$\langle 3 \rangle = \{ 3^k \mid k \in \mathbb{Z} \}$

$= \{ \ldots, 3^{-4}, 3^{-3}, 3^{-2}, 3^{-1}, 1, 3, 3^2, 3^3, 3^4 \}$

$= \{ \ldots, \frac{1}{3^4}, \frac{1}{3^3}, \frac{1}{3^2}, \frac{1}{3}, 1, 3, 3^2, 3^3, 3^4, \ldots \}$

7 (a)

$$\det(S) = \det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 \cdot 0 - (-1)(1) = 1 \neq 0$$

So, $S \in GL(2, \mathbb{R})$

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7 (b)

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S^3 = S \cdot S^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$S^4 = S \cdot S^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, S has order 4 and

$$\langle S \rangle = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{identity element } I}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

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8 Note $\det(T) = \det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 - 0 = 1$. So, $T \in GL(2, \mathbb{R})$.

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$T^3 = T \cdot T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$T^4 = T \cdot T^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$\vdots \qquad \vdots$$

The pattern is $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ when $n \geq 1$.

$T^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So, $T^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$T^{-1} = \frac{1}{\det(T)}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$T^{-2} = T^{-1}T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$T^{-3} = T^{-1}T^{-2} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

$$T^{-4} = T^{-1}T^{-3} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

$$\vdots \qquad \vdots$$

The pattern is $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ when $n < 0$.

Thus,

$$\langle T \rangle = \left\{ \ldots, \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \ldots \right\}$$

$$= \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$$

(9) $D_6 = \{1, r, r^2, s, sr, sr^2\}$

and $r^3 = 1$, $s^2 = 1$.

Since $r^3 = 1$ we know $r^{-1} = r^3 r^{-1} = r^2$

Let $H = \{1, s, sr, sr^2\}$ in D_6.

We use a table to show that H is not a subgroup of D_6.

H	1	s	sr	sr^2
1	1	s	sr	sr^2
s	s	1		
sr	sr	r^2	1	
sr^2	sr^2			1

We can stop filling in the table. We see that

$$(sr)(s) = s\,rs = s\,sr^{-1}$$
$$= 1r^{-1} = r^2 \notin H$$

Since H is not closed under the operation, H is not a subgroup of D_6.

10 $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

and $r^4 = 1$, $s^2 = 1$, $r^k s = s r^{-k}$.

Let $H = \{1, r^2, s, sr^2\}$

We use a table to show that H is a subgroup of D_8.

H	1	r^2	s	sr^2
1	1	r^2	s	sr^2
r^2	r^2	1	sr^2	s
s	s	sr^2	1	r^2
sr^2	sr^2	s	r^2	1

Example calculations:

$$r^2 s = s r^{-2} = s r^4 r^{-2} = s r^2$$

$\boxed{r^4 = 1}$

$$r^2 (s r^2) = s r^{-2} r^2 = s$$

$$(s r^2) s = s r^2 s = s s r^{-2} = r^2$$

$$(sr^2)(sr^2) = s r^2 s r^2 = s s r^{-2} r^2$$

$$= s^2 \cdot 1 = 1$$

① $1 \in H$

② H is closed under the group operation by the table

③ H is closed under inversion by the table since $(r^2)^{-1} = r^2 \in H$, $s^{-1} = s \in H$, $(sr^2)^{-1} = sr^2 \in H$

By ①, ②, ③ we have that $H \trianglelefteq D_8$

(11) Let $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$

<u>proof that $N \trianglelefteq GL(2,\mathbb{R})$</u> :

(1) Setting $x = 0$ gives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in N$

(2) Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ be in N where $a, b \in \mathbb{R}$.

Then,

$$AB = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$$

which satisfies $a+b \in \mathbb{R}$.

So, $AB \in N$.

(3) Let $C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in N$ where $c \in \mathbb{R}$.

Then, $C^{-1} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \in N$ because $-c \in \mathbb{R}$.

By (1), (2), (3) we know $N \trianglelefteq GL(2,\mathbb{R})$

$\blacksquare$

$\boxed{12}$ Let $H = \{2x+3y \mid x,y \in \mathbb{Z}\}$

proof that $H \trianglelefteq \mathbb{Z}$:

① Setting $x=0, y=0$ gives $0 = 2(0)+3(0) \in H$

② Let $a = 2x_1+3y_1$ and $b = 2x_2+3y_2$ be in H where $x_1, y_1, x_2, y_2 \in \mathbb{Z}$.

Then,

$$\begin{aligned} a+b &= 2x_1+3y_1+2x_2+3y_2 \\ &= 2(x_1+x_2)+3(y_1+y_2) \end{aligned}$$

is in H since $x_1+x_2, y_1+y_2 \in \mathbb{Z}$.

③ Let $c = 2x_3+3y_3$ be in H where $x_3, y_3 \in \mathbb{Z}$.

Then, $-c = 2(-x_3)+3(-y_3)$ is in H since $-x_3, -y_3 \in \mathbb{Z}$.

By ①, ②, ③ we have $H \trianglelefteq \mathbb{Z}$. ▨

$\boxed{13}$ $H = \{x \in G \mid x^2 = e\}$ and G is an abelian group.

proof that $H \trianglelefteq G$:

(1) $e^2 = e$ gives that $e \in H$.

(2) Let $a, b \in H$.

Then $a^2 = e$ and $b^2 = e$.

Thus,

$$
\begin{aligned}
(ab)^2 = (ab)(ab) &= abab \\
&= aabb \\
&= a^2 b^2 \\
&= ee = e
\end{aligned}
$$

since G is abelian

Since $(ab)^2 = e$ we get that $ab \in H$

(3) Let $c \in H$.

Then $c^2 = e$.

So, $c^{-2} c^2 = c^{-2} e$

Thus, $e = c^{-2}$.

So, $e = (c^{-1})^2$

Thus, $c^{-1} \in H$.

By (1), (2), (3) we have that $H \trianglelefteq G$. ▨

14

1. Since $H \trianglelefteq G$ we know $e \in H$.
   Since $K \trianglelefteq G$ we know $e \in K$.
   Thus, $e \in H \cap K$.

2. Let $a, b \in H \cap K$.
   So, $a \in H \cap K$ and $b \in H \cap K$.
   Thus, $a \in H, a \in K, b \in H, b \in K$.
   Since $H \trianglelefteq G$ and $a, b \in H$ we know $ab \in H$.
   Since $K \trianglelefteq G$ and $a, b \in K$ we know $ab \in K$.
   Thus, $ab \in H \cap K$.

3. Let $c \in H \cap K$.
   Then $c \in H$ and $c \in K$.
   Since $H \trianglelefteq G$ and $c \in H$ we know $c^{-1} \in H$.
   Since $K \trianglelefteq G$ and $c \in K$ we know $c^{-1} \in K$.
   Thus, $c^{-1} \in H \cap K$.

By 1, 2, 3 we know that $H \cap K \trianglelefteq G$.

$\boxed{//}$

(15) G is abelian, $H \trianglelefteq G$, $K \trianglelefteq G$

$HK = \{hk \mid h \in H, k \in K\}$

proof that $HK \trianglelefteq G$:

① Since $H \trianglelefteq G$ we know $e \in H$.
Since $K \trianglelefteq G$ we know $e \in K$.
Thus, $e = ee \in HK$.

② Let $a, b \in HK$.
Then $a = h_1 k_1$ and $b = h_2 k_2$
where $h_1, h_2 \in H$ and $k_1, k_2 \in K$.

We have

$$ab = h_1 k_1 h_2 k_2 = h_1 h_2 k_1 k_2$$

Since G is abelian

Since $h_1, h_2 \in H$ and $H \trianglelefteq G$ we know $h_1 h_2 \in H$.
Since $k_1, k_2 \in K$ and $K \trianglelefteq G$ we know $k_1 k_2 \in K$
Thus, $ab = (h_1 h_2)(k_1 k_2) \in HK$.

③ Let $c \in HK$.
Then $c = hk$ where $h \in H$ and $k \in K$.
Since $h \in H$ and $H \trianglelefteq G$ we
know that $h^{-1} \in H$.

Since $k \in K$ and $K \trianglelefteq G$ we know that $k^{-1} \in K$.

formula from class

Thus,

$$c^{-1} = (hk)^{-1} = k^{-1}h^{-1}$$

since G is abelian

$$= h^{-1}k^{-1} \in HK$$

By ①, ②, ③ we know that $HK \trianglelefteq G$.

$\blacksquare$

16

1 We know that $ey = y = ye$ for all $y \in G$. Thus, $e \in Z(G)$.

2 Let $a, b \in Z(G)$.

Then $ay = ya$ for all $y \in G$
and $by = yb$ for all $y \in G$.

Thus,

$$(ab)y = aby \overset{by=yb}{=} ayb \underset{ay=yb}{=} yab = y(ab)$$

for all $y \in G$.

So, $ab \in Z(G)$.

3 Let $c \in Z(G)$.

Then, $cy = yc$ for all $y \in G$.

So, $c^{-1}(cy)c^{-1} = c^{-1}(yc)c^{-1}$ for all $y \in G$.

Thus, $yc^{-1} = c^{-1}y$ for all $y \in G$.

Hence $c^{-1} \in Z(G)$.

By 1, 2, 3 we know that $Z(G) \trianglelefteq G$.